1.

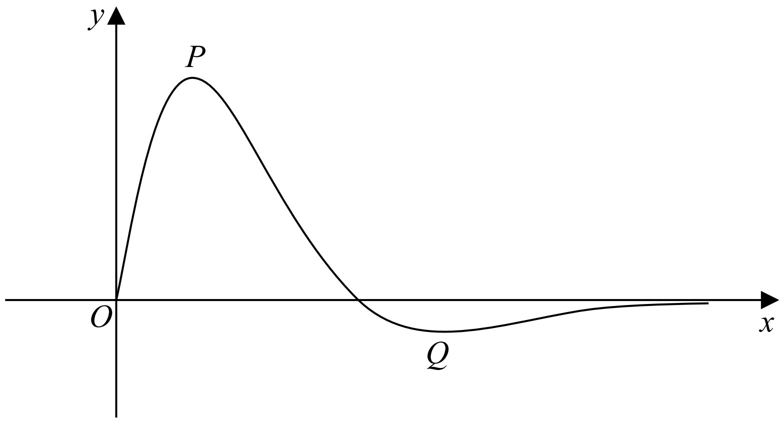


Figure 5

Figure 5 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{4\sin 2x}{e^{\sqrt{2}x-1}}, \quad 0 \leqslant x \leqslant \pi$$

The curve has a maximum turning point at P and a minimum turning point at Q as shown in Figure 5.

(a) Show that the x coordinates of point P and point Q are solutions of the equation

$$\tan 2x = \sqrt{2}$$

(4)

(b) Using your answer to part (a), find the x-coordinate of the minimum turning point on the curve with equation

(i) $y = \mathrm{f}(2x)$.

(ii) $y = 3 - 2\mathrm{f}(x)$.

(4)

Question continued

a) $f(x) = \dfrac{4\sin 2x}{e^{\sqrt{2}x-1}}$

Quotient Rule : 1

$f'(x) = \dfrac{h'(x)\cdot g(x) - h(x)\cdot g'(x)}{(g(x))^2}$

• Stationary Point when $f'(x) = 0$

let $h(x) = 4\sin 2x$ $\quad h'(x) = 8\cos 2x$

$g(x) = e^{\sqrt{2}x-1}$ $\quad g'(x) = \sqrt{2}e^{\sqrt{2}x-1}$

$\Rightarrow f'(x) = \dfrac{8\cos 2x\cdot e^{\sqrt{2}x-1} - 4\sin 2x\cdot\sqrt{2}e^{\sqrt{2}x-1}}{(e^{\sqrt{2}x-1})^2} = 0$ 1

$\Rightarrow 8\cos 2x\cdot e^{\sqrt{2}x-1} - 4\sqrt{2}\sin 2x e^{\sqrt{2}x-1} = 0$

$\Rightarrow e^{\sqrt{2}x-1}(8\cos 2x - 4\sqrt{2}\sin 2x) = 0$ 1

$\Rightarrow 8\cos 2x - 4\sqrt{2}\sin 2x = 0$

$\Rightarrow 8\cos 2x = 4\sqrt{2}\sin 2x$

$\Rightarrow 8 = \dfrac{4\sqrt{2}\sin 2x}{\cos 2x} \Rightarrow \dfrac{\sin 2x}{\cos 2x} = \dfrac{8}{4\sqrt{2}}$

$* \tan 2x = \dfrac{\sin 2x}{\cos 2x}$

$\Rightarrow \tan(2x) = \sqrt{2}$ as required 1

b) i) $y = f(2x)$

For $y = f(x) \Rightarrow \tan 2x = \sqrt{2}$

For $y = f(2x) \Rightarrow \tan 4x = \sqrt{2}$

$x = \dfrac{\tan^{-1}\sqrt{2}}{4} + \dfrac{\pi}{4}$ 1

$x = 1.024$ 1

ii) $y = 3 - 2f(x) \Rightarrow \tan 2x = \sqrt{2}$

$x = \dfrac{\tan^{-1}(\sqrt{2})}{2} = 0.478$ 1 1

2.

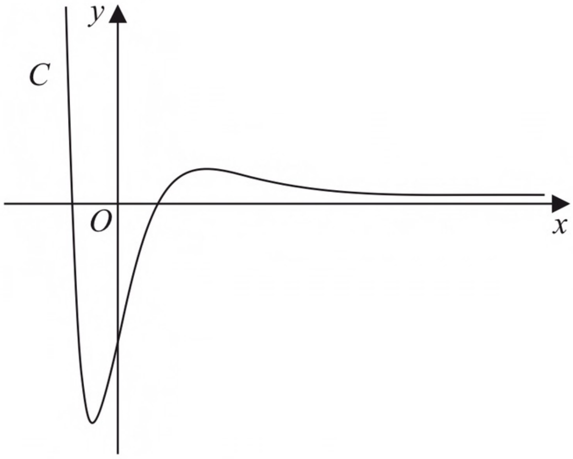


Figure 2

Figure 2 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 4(x^2 - 2)\mathrm{e}^{-2x} \qquad x \in \mathbb{R}$$

(a) Show that $\mathrm{f}'(x) = 8(2 + x - x^2)\mathrm{e}^{-2x}$

(3)

a) $f(x) = 4(x^2 - 2)e^{-2x}$

Product Rule

$f(x) = g(x) \cdot h(x)$ then $f'(x) = g'(x)h(x) + g(x)h'(x)$

let $g(x) = 4(x^2 - 2)$ then $g'(x) = 8x$

$h(x) = e^{-2x}$ then $h'(x) = -2e^{-2x}$ (1)

$\Rightarrow f'(x) = 8x \cdot e^{-2x} + 4(x^2-2) \cdot -2e^{-2x}$ (1)

$= 8x \cdot e^{-2x} - 8e^{-2x}(x^2 - 2)$

$f'(x) = 8(x - x^2 + 2)e^{-2x}$

$\Rightarrow f'(x) = 8(2 + x - x^2)e^{-2x}$ as required. (1)

(b) Hence find, in simplest form, the exact coordinates of the stationary points of C.

(3)

b) $f'(x) = 8(2+x-x^2)e^{-2x}$

Stationary Points : $f'(x) = 0$

$\Rightarrow \quad 8(2+x-x^2)e^{-2x} = 0$ $\quad \div 8$ and $\div e^{-2x}$ (on both sides)

$\Rightarrow \quad 2+x-x^2 = 0$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x-2)(x+1) = 0$

M	A
-2	-1

$-2+1 = -1$ ✓

$\Rightarrow \quad x = 2$ and $x = -1$ ①

For $x = 2$, $y = f(2) = 4((2)^2-2)e^{-2(2)} = 4(2)e^{-4} = 8e^{-4} = y$ ①

For $x = -1$, $y = f(-1) = 4((-1)^2-2)e^{-2(-1)} = -4e^2 = y$

$\Rightarrow$ Our coordinates are : $\underline{(2, 8e^{-4})}$ and $\underline{(-1, -4e^2)}$ ①

The function g and the function h are defined by

$$g(x) = 2f(x) \qquad x \in \mathbb{R}$$

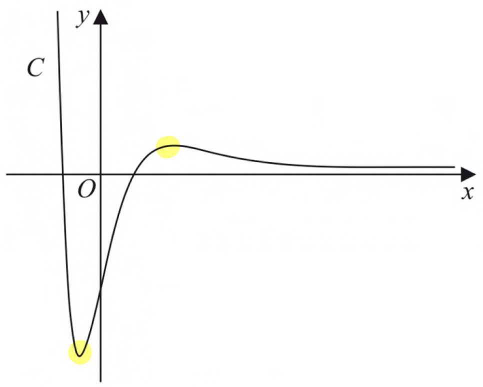
$$h(x) = 2f(x) - 3 \qquad x \geqslant 0$$

(c) Find (i) the range of g

(ii) the range of h

(3)

c) i) $f(x) = 4(x^2-2)e^{-2x} \Rightarrow g(x) = 2f(x) = 8(x^2-2)e^{-2x}$



If coordinates of $f(x) : (a, b)$ then $g(x) : (a, 2b)$

lower limit of range: $2 \times -4e^2 = -8e^2$
upper limit of range: ∞

$\Rightarrow$ Range : $[-8e^2, \infty)$ 1

c) ii) $h(x) = 2f(x) - 3 = 8(x^2-2)e^{-2x} - 3$ for $x \geqslant 0$

The lower limit of the range will be at $x = 0 \Rightarrow h(0) = 8(-2)e^{-2\times 0} - 3$
$\Rightarrow h(0) = -19$ 1

The upper bound will be our maximum turning point (Since $x \geqslant 0$).
From part b this max turning point had y-value of $8e^{-4}$.
$\Rightarrow$ For the $h(x)$ function this point will be $2 \times 8e^{-4} - 3 = 16e^{-4} - 3$

Range : $[-19, 16e^{-4} - 3]$ 1

3. A scientist is studying a population of mice on an island.

The number of mice, N, in the population, t months after the start of the study, is modelled by the equation

$$N = \frac{900}{3 + 7e^{-0.25t}}, \quad t \in \mathbb{R}, \quad t \geqslant 0$$

(a) Find the number of mice in the population at the start of the study.

$\hookrightarrow t = 0$

(1)

(b) Show that the rate of growth $\frac{dN}{dt}$ is given by $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$

(4)

The rate of growth is a maximum after T months. $\rightarrow \frac{dN}{dt} = 0$

(c) Find, according to the model, the value of T.

(4)

According to the model, the maximum number of mice on the island is P.

(d) State the value of P. $\hookrightarrow t \rightarrow \infty$

(1)

Question continued

a) Start $\Rightarrow t = 0$

$$N_{(start)} = \frac{900}{3 + 7e^{-0.25(0)}}$$

$$= \frac{900}{10} = 90$$

Question continued

b)

$$N = \frac{900}{3+7e^{-0.25t}}$$

$$N = 900(3+7e^{-0.25t})^{-1}$$

$$y = f(g(t)) \Rightarrow y = f(u) \quad u = g(t)$$

$$\frac{dy}{dx} = f'(u) \times g'(t)$$

$$N = 900u^{-1} \qquad u = 3+7e^{-0.25t}$$

$$\frac{dN}{du} = -900u^{-2} \qquad \frac{du}{dt} = \cancel{0} - 0.25 \times 7 \times e^{-0.25t}$$

$$\frac{dN}{dt} = 900 \times 0.25 \times 7e^{-0.25t} \times (3+7e^{-0.25t})^{-2}$$

$$N = \frac{900}{3+7e^{-0.25t}}, \quad 3+7e^{-0.25t} = \frac{900}{N}$$

$$7e^{-0.25t} = \frac{900}{N} - 3$$

$$\frac{dN}{dt} = \frac{900}{4} \times \left(\frac{900}{N} - 3\right) \times \left(\frac{N^2}{900^2}\right)$$

$$= \frac{\cancel{900}}{4} \times \frac{\cancel{3}}{\cancel{N}}(300-N) \times \frac{\cancel{N^2}\ N}{\cancel{900^2}\ \cancel{900}\ 300}$$

$$= \frac{N}{300} \times \frac{1}{4} \times (300-N)$$

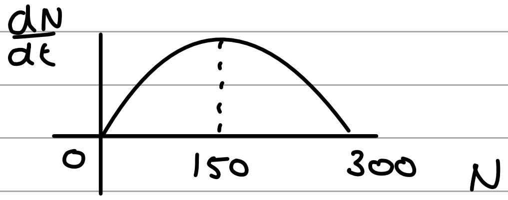
$$\therefore \frac{dN}{dt} = \frac{N(300-N)}{1200}$$

Question continued

c)

$$\frac{dN}{dt} = \frac{N(300-N)}{1200} \qquad N = \frac{900}{3+7e^{-0.25t}}$$

$$N(300-N) = 0$$



$\frac{dN}{dt}$ is maximum at $N = 150$

$$150 = \frac{900}{3+7e^{-0.25T}}$$

$$3+7e^{-0.25T} = 6$$

$$7e^{-0.25T} = 3$$

$$e^{-0.25T} = \frac{3}{7}$$

$$-0.25T = \ln\left(\frac{3}{7}\right)$$

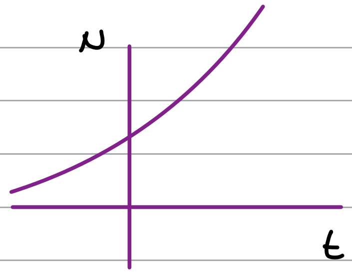
$$T = \ln\left(\frac{3}{7}\right) \times -4$$

$T = 3.38\ldots$ Months.

$T = 3.4$ Months

d)

$N = \dfrac{900}{3+7e^{-0.25t}}$



as $t \to \infty$, $0.25t \to \infty$, $-0.25t \to -\infty$

$e^{-0.25t} \to 0$

$P = \dfrac{900}{3+7(0)} = \dfrac{900}{3} = 300$ ✓

4. The curve C has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$$

(4)

a) We want to use implicit differentiation to differentiate $x^2 \tan y = 9$

$x^2 \longrightarrow 2x$

$\tan y \longrightarrow \sec^2 y \frac{dy}{dx}$

Product Rule

$h(x) = f(x) \cdot g(x)$ then

$h'(x) = f'(x)\, g(x) + f(x)\, g'(x)$

$\Rightarrow \quad 2x \cdot \tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$ (2)

1 for attempting to differentiate

1 for correct differentiation

We will use the trig identity : $\sec^2 y = 1 + \tan^2 y$ and $\tan y = \frac{9}{x^2}$

$\boxed{\Rightarrow \tan^2 y = \frac{81}{x^4}}$

$\Rightarrow \quad 2x \cdot \frac{9}{x^2} + x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{18}{x} + x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = 0$

$\Rightarrow \quad x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = -\frac{18}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-\frac{18}{x}}{x^2\left(1 + \frac{81}{x^4}\right)}$ (1) $= \frac{-18}{x^3\left(1 + \frac{81}{x^4}\right)}$ *

* $x^3\left(1 + \frac{81}{x^4}\right) = x^3\left(\frac{x^4 + 81}{x^4}\right) = \frac{x^4 + 81}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-18}{\frac{x^4+81}{x}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-18x}{x^4 + 81}$ as required. (1)

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27} = (27)^{1/4}$

(3)

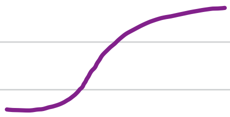
Quotient Rule :

$f(x) = \dfrac{h(x)}{g(x)}$ then

$f'(x) = \dfrac{h'(x)\cdot g(x) - h(x)\cdot g'(x)}{(g(x))^2}$

b) Part a : $\dfrac{dy}{dx} = \dfrac{-18x}{x^4+81}$

Point of inflection :



$-18x \longrightarrow -18$

$x^4+81 \longrightarrow 4x^3$ (1)

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-18(x^4+81) - 4x^3(-18x)}{(x^4+81)^2}$

$= \dfrac{-18x^4 - 1458 + 72x^4}{(x^4+81)^2}$

$= \dfrac{54x^4 - 1458}{(x^4+81)^2} = \dfrac{54(x^4-27)}{(x^4+81)^2} = \dfrac{d^2y}{dx^2}$ (1)

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{54(x^4-27)}{(x^4+81)^2}$

- At $x = \sqrt[4]{27} \Rightarrow x^4 = 27 \Rightarrow$ we can substitute this into $\dfrac{d^2y}{dx^2}$

$\Rightarrow$ For $x^4 = 27$, $\dfrac{d^2y}{dx^2} = \dfrac{54(27-27)}{(27+81)^2} = 0$

$\Rightarrow$ For $x^4 > 27$, $\dfrac{d^2y}{dx^2} > 0$

$\Rightarrow$ For $x^4 < 27$, $\dfrac{d^2y}{dx^2} < 0$

$\Rightarrow$ From this we can conclude that there is a point of inflection at $x = \sqrt[4]{27}$. (1)

5.

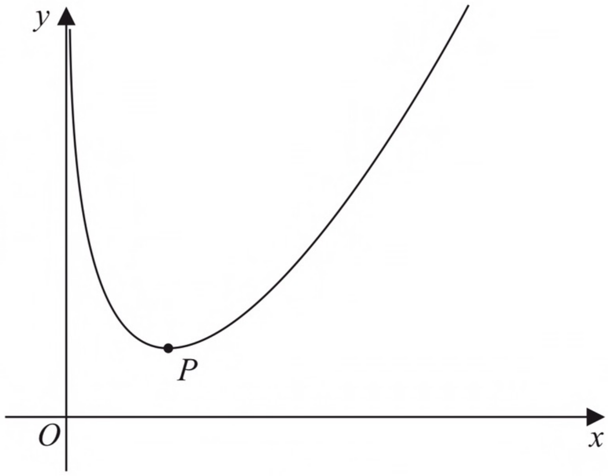


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

(4)

a) $y = \frac{4x^2+x}{2\sqrt{x}} - 4\ln x$, Find $\frac{dy}{dx}$

• Log Differentiation : $\frac{d}{dx}(\ln x) = \frac{1}{x}$

• Quotient Rule : If $h(x) = \frac{f(x)}{g(x)}$

then $h'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$

• $\frac{d}{dx}(4\ln x) = 4 \cdot \frac{1}{x} = \boxed{\frac{4}{x}}$ 1

let $h(x) = \frac{4x^2+x}{2\sqrt{x}} \Rightarrow f(x) = 4x^2 + x \rightarrow f'(x) = 8x + 1$

$2\sqrt{x} = 2x^{1/2}$

$g(x) = 2\sqrt{x} \rightarrow g'(x) = \frac{1}{\sqrt{x}}$ 1

$\Rightarrow h'(x) = \frac{(8x+1)(2\sqrt{x}) - (4x^2+x)\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x})^2}$ 1 $= \frac{16x^{3/2} + 2x^{1/2} - \frac{4x^2}{x^{1/2}} - \frac{x}{x^{1/2}}}{4x} = \frac{16x^{3/2} + 2x^{1/2} - 4x^{3/2} - x^{1/2}}{4x}$

$\Rightarrow h'(x) = \frac{12x^{3/2} + x^{1/2}}{4x} = 3x^{1/2} + \frac{1}{4x^{1/2}} = \boxed{3\sqrt{x} + \frac{1}{4\sqrt{x}}}$

$\Rightarrow \frac{dy}{dx} = 3\sqrt{x} + \frac{1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x+1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} = \frac{dy}{dx}$ as required. 1

The point P, shown in Figure 1, is the minimum turning point on C.

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

(3)

b) From part a : $\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$

Our first step is to set $\frac{dy}{dx} = 0$. $\Rightarrow \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} = 0$

$\Rightarrow 12x^2 + x - 16\sqrt{x} = 0$ $\quad \div \sqrt{x}$

$\Rightarrow 12x^{3/2} + \sqrt{x} - 16 = 0$ 1

$\Rightarrow 12x^{3/2} = 16 - \sqrt{x}$ $\quad \div 12$ 1

$\Rightarrow x^{3/2} = \frac{16}{12} - \frac{\sqrt{x}}{12}$

$\Rightarrow x^{3/2} = \frac{4}{3} - \frac{\sqrt{x}}{12}$

$\Rightarrow x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{2/3}$ as required. 1

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{\frac{2}{3}} \qquad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,

(ii) the x coordinate of P to 5 decimal places.

(3)

c)

i) $x_1 = 2$ and $x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{2/3} \Rightarrow x_2 = \left(\frac{4}{3} - \frac{\sqrt{x_1}}{12}\right)^{2/3} = \left(\frac{4}{3} - \frac{\sqrt{2}}{12}\right)^{2/3}$ 1

Sub this in!

$x_2 = 1.138935...$

$x_2 = 1.13894$ (5 d.p) 1

ii) $x = 1.15650$ 1